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B.Sc Part III - Date - 24-04-2020

Differentiability and Differentials

* **Introduction** :→ If $u = f(x,y)$ be a real function of two variables x and y. If (a,b) is any point of the domain of a function $u = f(x,y)$.

Then

$$\lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$ is called the partial derivative of $f(x,y)$ with respect to x at (a,b) Treating y as a constant and is denoted by $\frac{\partial u}{\partial x}$ or $f_x(a,b)$ provided the limit exist.

Similarly $\frac{\partial u}{\partial y}$ or $f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(b)}{k}$ provided limit exist.

* **Definition of differentiability and differential.**

Definition :→ A function $u = f(x,y)$ is said to be differentiable at a point (x,y) of its domain if the increment δu in the function corresponding to any arbitrary assigned increments δx and δy in x and y respectively, is expressed in the form

$$\delta u = f(x+\delta x, y+\delta y) - f(x,y)$$

$$= A\delta x + B\delta y + \delta x\, \varphi(\delta x, \delta y) + \delta y\, \psi(\delta x, \delta y) \quad \text{——— (1)}$$

Here A, B are constants independent of δx and δy and $\varphi(\delta x, \delta y)$, $\psi(\delta x, \delta y)$ are functions of $\delta x, \delta y$

Where

$$\lim_{(\delta x, \delta y) \to (0,0)} \varphi(\delta x, \delta y) = 0 = \lim_{(\delta x, \delta y) \to (0,0)} \psi(\delta x, \delta y)$$

Here $A\delta x + B\delta y$ is the linear part in δx and δy and is called the differential of $f(x,y)$ and (x,y) and it is denoted by du

(5) Then $du = A\delta x + B\delta y$ ——— (2)

If in the Equation (1), we regard y as fixed i.e $\delta y \to 0$ and x as variable, then

we have $\delta u = A\delta x + \delta x\, g(\delta x, 0)$

$$\frac{\delta u}{\delta x} = A + g(\delta x, 0)$$

$$\therefore A = \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad \text{since } g \to 0$$

$$= \frac{\partial u}{\partial x} \quad \text{since } y \text{ is regarded as constant}$$

Similarly if we regard x as fixed and y as variable, then we get

$$B = \frac{\partial u}{\partial y}$$

Putting the values of A and B in (2)

$$du = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$$

Thus if $u = f(x,y)$ is differentiable at (x,y) then the function must possess the partial derivatives f_x and f_y at the point (x,y)

It is the necessary condition for differentiability of a function $f(x,y)$ at a given point, but is not sufficient.

Show that if the function $f(x,y)$ is differentiable at a point (a,b) in its domain, then the partial derivatives f_x and f_y both exist and are finite —

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Sol. Since the function is differentiable at a point (a,b).

therefore

$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\,\varphi(h,k) + k\,\psi(h,k) \quad (1)$$

where A and B are constants independent of h and k, and $\varphi(h,k)$, $\psi(h,k)$ are functions of h and k so that

$$\lim_{(h,k)\to(0,0)} \varphi(h,k) = \lim_{(h,k)\to(0,0)} \psi(h,k)$$

Now putting $k=0$ and passing to the limit as $h\to 0$, we find from (1) that

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = A$$

i.e. $f_x(a,b) = A$

Similarly $f_y(a,b) = B$

[If the function is differentiable at a point (a,b) of its domain, then it is also continuous at (a,b)]

But if the function $f(x,y)$ is continuous at a point (a,b) of its domain then it may or it may not be differentiable at (a,b)]

Since.

1. Examples of functions which possess partial derivatives but are not continuous

2. Examples of functions which possess partial derivatives but not differentiable.

Ex.

* If $f(x,y) = 0$ when either x or y is 0 and
$f(x,y) = 1$ for $x \neq 0$, $y \neq 0$

Show that the function is discontinuous at the point $(0,0)$ but that the partial derivatives exist at $(0,0)$

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By the question,

$f(x,y)=0$ when Either $x=0$ or $y=0$

$f(x,y)=1$ when $x\neq 0$, $y\neq 0$

We have $|f(x,y)-f(0,0)|=1$ which is not $<\varepsilon$
it shows that $f(x,y)$ is Discontinuous at $(0,0)$

Again $f_x(0,0)=\lim\limits_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}$

$=\lim\limits_{h\to 0}\dfrac{0}{h}=0$

$f_y(0,0)=\lim\limits_{k\to 0}\dfrac{f(0,k)-f(0,0)}{k}$

$\lim\limits_{k\to 0}\dfrac{0}{k}=0$

Here we see Partial derivative $f_x(0,0)$ and $f_y(0,0)$ Exist but $f(x,y)$ is Discontinuous at $(0,0)$